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Double Barrier Cash or Nothing Options: a short note

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May 2009
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Abstract

In this note we discuss and summarize the valuation methodology for Double Barrier Cash or Nothing Options. We start off by briefly defining vanilla binary options and ordinary and double barrier options. We then move on to the valuation and price dynamics of the option at hand. After that we list the formulas for the Greeks and discuss their dynamics. Lastly we take a look at the asymptotes if one of the barriers disappears or falls away.

‡We want to thank the JSE (especially Safex) in Johannesburg, South Africa for their assistance in the preparation of this note.

1 Binary Options

A binary option is a type of option where the payoff is discontinuous in the underlying asset – it is either some fixed amount of some asset or nothing at all[1]. The two main types of binary options are the cash-or-nothing and the asset-or-nothing options [Wi98]. The cash-or-nothing option pays some fixed amount of cash if the option expires in-the-money whilst the asset-or-nothing pays the value of the underlying security. The payoffs of these options are binary in nature because there can only be two possible outcomes [RR91b]. These options are also known as digital options.

Binary options are used as a hedge against "jump risk" e.g., where the underlying asset is illiquid and as such the price can suddenly jump to another level. They are generally simpler to trade because they require only a sense of direction of the price movement of the underlying asset, whereas traditional options require a sense of direction as well as the magnitude of the price movement. The biggest advantage of binary options is that they have a controlled risk to reward ratio, meaning the risk and reward are predetermined at the time the contract is acquired. Due to this they are widely used in the sports betting industry.

2 Barrier Options

European, continuously-monitored barrier options are European options with an American feature. Option's existence depends on whether the underlying price breaches, before or at maturity, some pre-specified level, called a barrier [Sb00]. These options are standard calls and put except that they either disappear (the option is knocked out) or appear (the option is knocked in) if the underlying asset price breach a predetermined level (the barrier) [RR91a]. Barrier options are thus conditional options, dependent on whether the barriers have been crossed within the lives of the optionscitekn:Ko98. These options are also part of a class of options called path-dependent options[2,3].

Barrier options are usually cheaper than their vanilla counterparts [Ya03]. This is due to the fact that a buyer of a barrier option has a more specific

[1]The term "binary" comes from the computer or mathematics jargon: a binary number is one which is given a value of either 0 or 1 and nothing else. Applying that terminology, a binary option is an option that, at expiration, can have only one of two payoff possibilities, a 0 or a 1.

[2]A path-dependent option is an option whose payoff depends on the history of the underlying asset price.

[3]Other path-dependent options are Asian options, look-back options, ladder options and chooser or shout options.

view of the underlying asset price dynamics within the time to maturity of the option [Zh98]. These options are also called single barrier options. Note that a portfolio of a knock-in and a knock-out written on the same barrier and strike is equivalent to a vanilla option with the same strike.

Another hybrid to the barrier family is the so-called partial time barrier option. Here, the barrier is monitored (or active) for a time period that is shorter than the expiry time [St07]. These options are also called window barrier options [Zh98].

Another refinement is where the barrier is monitored discretely in time. Most traded barriers are monitored daily using the official close price of the underlying asset. Broadie, Glasserman and Kou studied this and solved the problem. They connected the continuous- and discrete time monitored barrier options by making an adjustment to the barrier level [BG99].

Another style of barrier option is the double barrier. Here there is both an upper and a lower barrier, the first above and second below the current asset price. Double knock-ins come to life and double knockouts terminate if either barrier is hit [Ha07].

Via double barriers, investors enjoy even greater leverage potential: single knock-outs typically have barriers too close for comfort and single knock-ins have less knock-in chances without much discount. A double knock-in may be bought by a fund manager who bets against market consensus' direction

but hedges her bet for marking-to-market purposes. It may also be bought by a trader who foresees a bigger volatility than the market consensus' one in both bullish and bearish scenarios [Sb00].

3 Double Barrier Binary Options

A step further along the option evolution path is where we combine barrier and binary options to obtain "binary barrier options". There are 28 variants [RR91a, Ha07]. Taking this another step further, we combine a double barrier with a binary option. These double barrier binary options are not combinations of single barrier binary options. Hui discussed the pricing of the cash or nothing variants in a Black & Scholes environment [Hu96]. There are only two types: knock-outs and knock-ins.

3.1 Option Valuation

Definition 3.1 *Double Barrier Cash or Nothing Knock-Out (dbcnko) option:*
The option pays zero (expires worthless) if the underlying price S touches either the lower barrier B_L or the upper barrier B_U, during the lifetime T of the option. However, it pays a pre-defined cash amount R at expiry if neither of the barriers are hit during the lifetime of the option.

♠

This option is quite simple because we do not have to specify a strike price - the barriers act as the triggers whether it will pay out or not.

Intuitively, we can claim that the value of the knock-out option is the probability of the underlying price S staying within the barriers during the option's lifetime. The upper and lower barriers define a boundary range. Using the Fourier sine series[4], we can show that the risk-neutral value of a Double Barrier Cash or Nothing Knock-Out is [Hu96, Ha07]

$$\begin{aligned} V_{dbcnko} &= \sum_{i=1}^{\infty} \frac{2\pi\, i\, R}{Z^2} \left[\frac{\left(\frac{S}{B_L}\right)^{\alpha} + (-1)^{i+1} \left(\frac{S}{B_U}\right)^{\alpha}}{\alpha^2 + \left(\frac{i\pi}{Z}\right)^2} \right] \\ &\times \ \sin\left(\frac{i\pi}{Z} \ln\left(\frac{S}{B_L}\right)\right) \times \ e^{-\frac{1}{2}\left[\left(\frac{i\pi}{Z}\right)^2 - \beta\right]\sigma^2 T} \end{aligned} \tag{1}$$

where

$$Z = \ln\left(\frac{B_U}{B_L}\right), \qquad \alpha = \frac{d-r}{\sigma^2} + \frac{1}{2}, \qquad \beta = -\frac{1}{4}\left(\frac{2\,(r-d)}{\sigma^2} - 1\right)^2 - 2\frac{r}{\sigma^2}.$$

We define r to be the risk-free rate in continuous format, d the dividend yield in continuous format and σ is the annualised underlying's price volatility.

In Figure 3.1 we give a VBA function to implement this option in Excel.

Hui showed that the series in Eq. (1) converges quite quickly. Our tests show that $n = 100$ gives good results in all circumstances: where the barriers are close to the spot price and when T is very small. Table (3.1) shows the dynamics of the option price with respect to S and time to expiry. This clearly shows that the option value diminishes the closer we are to one of the barriers but it gyrates towards the cash payout R when we are close to expiry and far from the barriers.

[4]The sine trigonometric is a periodic function with waves bounded by the amplitude. It therefore makes sense to use the Fourier sine series in the pricing of these options.

```
Function double_cash_nothing(S As Double, Bu As Double, Bl As Double, _
                             Rebate As Double, r As Double, _
                             sigma As Double, T As Double, _
                             d As Double) As Double

' Knock-Out: if either barrier is hit, pays nothing. If not, pays rebate
' Knock-in: if either barrier is hit, pays rebate, if not, pays nothing
' S = spot price
' Bu = upper barrier; Bl = lower barrier
' r = interest rate in continuous format
' d = dividend yield in continuous format
' sigma = volatility
' T = annualised time to expiry
' Rebate = cash to pay out
Pi=3.14159265358979
If (T < 0) Then
  SOM = 0
ElseIf S >= Bu Or S <= Bl Then
' Option knocked out
  SOM = 0
Else
  Alfa = -0.5 * (2 * (rd - d) / sigma / sigma - 1)
  Beta = -0.25 * (2 * (rd - d) / sigma / sigma - 1) ^ 2 - _
         2 * rd / sigma / sigma
  Z = Log(Bu / Bl)
  SOM = 0
  For i = 1 To 100
    SOM = SOM + 2 * Pi * i * Rebate / Z / Z _
          * (((S / Bl) ^ Alfa - (-1) ^ i * (S / Bu) ^ Alfa) / _
          (Alfa * Alfa + (i * Pi / Z) ^ 2)) _
          * Sin(i * Pi / Z * Log(S / Bl)) * _
          Exp(-0.5 * ((i * Pi / Z) ^ 2 - Beta) * sigma * sigma * T)
  Next
End If
double_cash_nothing = SOM
End Function
```

Figure 1: VBA code to implement Eq. (1).

Time\ Spot	85.5	90	92.5	95	97.5	100	102.5	105	107.5	110	114.5
0.5041	2.64	24.34	33.51	39.86	43.14	43.33	40.63	35.43	28.21	19.51	1.98
0.4192	4.67	42.96	59.16	70.38	76.16	76.49	71.74	62.56	49.80	34.44	3.50
0.3370	8.09	74.47	102.55	121.99	132.02	132.59	124.36	108.44	86.32	59.71	6.07
0.2521	14.29	131.49	181.05	215.38	233.08	234.09	219.55	191.44	152.40	105.41	10.72
0.1671	25.23	232.19	319.68	380.27	411.49	413.27	387.58	337.96	269.04	186.09	18.93
0.0849	44.25	405.32	555.68	658.52	710.96	713.82	670.51	586.37	468.39	324.97	33.14
0.0027	250.95	997.98	999.79	999.79	999.79	999.79	999.79	999.79	999.56	984.57	188.08

Table 1: Price dynamics of an option with $T = 0.5041$. The stock price is 100, the payout R1000, the volatility 35%, the riskfree rate 8% (naca) and dividend yield is 2% (naca).

The knock-in counterpart can be valued as a short Double Barrier Cash or Nothing Knock-Out plus the discounted cash payout R [Ha07]. It is given by

$$V_{dbcnki} = R\, e^{-rT} - V_{dbcnko} \tag{2}$$

3.2 Hedge Parameters

The delta of V_{dbcnko} is obtained by taking the first derivative of Eq. (1) with respect to the underlying price S such that

$$\begin{aligned}
\Delta_{dbcnko} &= \frac{\partial V_{dbcnko}}{\partial S} \\
&= \sum_{i=1}^{\infty} \frac{2\pi\, i\, R}{S\, Z^2} \left[\frac{\left(\frac{S}{B_L}\right)^{\alpha} + (-1)^{i+1} \left(\frac{S}{B_U}\right)^{\alpha}}{\alpha^2 + \left(\frac{i\pi}{Z}\right)^2} \right] \left[\frac{\frac{i\pi}{Z}}{\tan\left(\frac{i\pi}{Z} \ln \frac{S}{B_L}\right)} + \alpha \right] \\
&\times \ \sin\left(\frac{i\pi}{Z} \ln \frac{S}{B_L}\right) e^{-\frac{1}{2}\left[\left(\frac{i\pi}{Z}\right)^2 - \beta\right] \sigma^2 T}
\end{aligned} \tag{3}$$

Table (3.2) shows the dynamics of the delta with respect to the underlying price S and time to maturity for the same option shown in Table (3.1). We calculate the Vega (volatility risk) numerically by bumping the volatility up by 1%. The Vega dynamics for the same option shown above is given in Table 3.2.

The risk parameters for the Knock-In can be obtained from Eq. (2) such that

$$\begin{aligned}
\Delta_{dbcnki} &= -\Delta_{dbcnko} \\
\Gamma_{dbcnki} &= -\Gamma_{dbcnko} \\
Vega_{dbcnki} &= -Vega_{dbcnko}.
\end{aligned}$$

Time\ Spot	85.5	90	92.5	95	97.5	100	102.5	105	107.5	110	114.5
0.5041	5.27	4.17	3.13	1.93	0.69	-0.52	-1.61	-2.52	-3.22	-3.70	-3.97
0.4192	9.30	7.36	5.53	3.42	1.21	-0.92	-2.84	-4.45	-5.69	-6.53	-7.02
0.3370	16.12	12.76	9.58	5.92	2.10	-1.59	-4.92	-7.71	-9.86	-11.31	-12.16
0.2521	28.46	22.53	16.92	10.45	3.70	-2.81	-8.69	-13.62	-17.41	-19.97	-21.47
0.1671	50.26	39.78	29.87	18.44	6.53	-4.97	-15.34	-24.04	-30.74	-35.25	-37.90
0.0849	88.13	68.78	50.98	31.10	10.91	-8.39	-25.90	-40.94	-52.88	-61.22	-66.37
0.0027	483.64	3.73	0.01	0.00	0.00	-0.00	-0.00	-0.00	-0.46	-20.82	-369.87

Table 2: Dynamics of the Delta of an option with $T = 0.5041$. The stock price is 100, the payout R1000, the volatility 35%, the riskfree rate 8% (naca) and dividend yield is 2% (naca).

Time\ Spot	85.5	90	92.5	95	97.5	100	102.5	105	107.5	110	114.5
0.5041	-0.47	-4.33	-5.94	-7.04	-7.60	-7.61	-7.11	-6.18	-4.91	-3.39	-0.34
0.4192	-0.71	-6.47	-8.87	-10.51	-11.34	-11.34	-10.60	-9.21	-7.31	-5.04	-0.51
0.3370	-1.01	-9.19	-12.59	-14.90	-16.05	-16.04	-14.98	-13.00	-10.30	-7.10	-0.72
0.2521	-1.36	-12.40	-16.96	-20.05	-21.55	-21.51	-20.05	-17.38	-13.75	-9.45	-0.95
0.1671	-1.65	-14.93	-20.35	-23.96	-25.68	-25.54	-23.74	-20.52	-16.19	-11.10	-1.11
0.0849	-1.66	-14.33	-18.82	-21.42	-22.44	-22.16	-20.77	-18.29	-14.75	-10.30	-1.04
0.0027	-6.86	-0.61	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.11	-3.05	-5.04

Table 3: Dynamics of the Vega of an option with $T = 0.5041$. The stock price is 100, the payout R1000, the volatility 35%, the riskfree rate 8% (naca) and dividend yield is 2% (naca).

3.3 Asymptotes of the Double Barrier Binary Option

If the lower barrier, B_L is much lower than the underlying price S, the presence of that barrier becomes insignificant. The Double Barrier Cash or Nothing Knock-Out option then approaches the value of the single barrier Up-and-Out Cash-or-Nothing Put (kocn) with a rebate R with both barrier and strike at the upper barrier B_U. In short,

$$\lim_{B_L \to 0} V_{dbcnko} \cong V_{kocn}$$

with strike $K \approx B_U$.

The pricing formula for this option is due to Reiner and Rubinstein [RR91b] (also see [Ha07])

$$V_{kocn} = R\, e^{-r\tau} \left[N\left(\phi x_1 - \phi\sigma\sqrt{T}\right) - \left(\frac{B}{S}\right)^{2\mu} N\left(\eta y_1 - \eta\sigma\sqrt{T}\right) \right] \qquad (4)$$

where

$$x_1 = \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{S}{K}\right) + (\mu + 1)\sigma\sqrt{T}$$

$$
\begin{aligned}
y_1 &= \frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{B^2}{S\,K}\right) + (\mu+1)\sigma\sqrt{T} \\
\lambda &= \sqrt{\mu^2 + \frac{2\,r}{\sigma^2}} \\
\mu &= \frac{r-d}{\sigma^2} - \frac{1}{2}
\end{aligned}
$$

and $N(\bullet)$ is the cumulative normal distribution function (see [Ha07]), K is the strike price , B the barrier level and R is the cash payout. To price the Up-and-Out Cash-or-Nothing Put we must use

$$\phi = \eta = -1;\ \ B = K = B_U.$$

Values obtained from the formulas in Eqs (1) and (4) converge if the lower barrier level is very small in Eq (1) - a lower barrier 1% of the spot price, will suffice.

If the upper barrier is significantly higher than the underlying price S, the Double Barrier Cash or Nothing Knock-Out option approaches the value of the Down-and-Out Cash-or-Nothing Call (docn) with both barrier and strike at the lower barrier B_L. The value of this option is given by Eq. (4) where we have

$$\phi = \eta = 1;\ \ B = K = B_L$$

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