Pricing JSE Exotic Can-Do Options: Monte Carlo Simulation

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Abstract

Monte Carlo simulation or probability simulation is a technique used to understand the impact of risk and uncertainty in financial and other forecasting models. It is very useful when complex financial instruments need to be priced. Exotic options are listed on the JSE on its Can-Do platform. Most listed exotic options are marked-to-model and the JSE needs accurate values at the end of every day. Monte Carlo methods in a local volatility framework are used when exotic options are priced. This paper discusses Monte Carlo (MC) simulation as implemented and used by the JSE.

Keywords: Exotic options, JSE, Can-Do Options, Implied Volatility, Local Volatility, Dupire Transforms, Gyöngy Theorem, Barrier options, Monte Carlo simulation, Feynmann-Kac Theorem

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1. Introduction

Why is simulation an important component of analysis for scientists? Simulation is the imitation of a real-world process or system. The "joy of simulation" is that one does not need to own or rent a Boeing 777 or Airbus A380 to fly them! Simulation games are fun too and one gains valuable experience at the same time. Experience and insight are gained by simulating the valuation of financial products, constructing portfolios and testing trading rules (McLeish, 2005). Through simulation work is transferred to the computer. Models can be handled that involve greater complexity and fewer assumptions, and a more faithful representation of the real world is possible.

Scientists investigate the world around us by building models and analysing those. These models usually take the form of differential equations that have to be solved to obtain physical answers. They usually start with a very simplistic model and try to solve it analytically or algebraically. This inevitably means they have to make a lot of simplifying assumptions. As they start to understand the dynamics of this "toy" model, they add more complexity to make it more representative of the real world. This is exactly the route the evolution of the Black-Scholes option model took. Black, Scholes and Merton made a lot of simplifying assumptions that enabled them to solve this differential equation exactly. This model is, however, far from the truth and as some of these assumptions are relaxed, one finds that the model cannot be solved analytically anymore. This should not discourage anyone because simulation techniques are usually great "complex problem solvers."

Monte Carlo simulation has become an essential tool in the pricing of derivative securities and the management of risk. Most problems where there is significant uncertainty, can be solved using Monte Carlo techniques. Monte Carlo methods are techniques utilising random numbers and probability to solve problems. The analysis is based on artificially recreating a chance process, running it many times and directly observing the results. Glassermann (2004) states it is thus based on the analogy between probability and volume.

Monte Carlo methods are attractive in evaluating integrals in high dimensions Glassermann (2004). What does this have to do with financial engineering? The foundation of the theory of derivative pricing is the random walk of asset prices. This is known as the Black-Scholes theory and leads to the Black-Scholes parabolic partial differential equation (PDE). According to the Feynman-Kac theorem, the solution to this PDE can be represented by an expected value — valuing derivatives is reduced to computing expectations. Monte Carlo simulation is widely used in statistics in calculating an expected value of a particular function. This thus found its way into finance where all options are always the expected value of certain functions (Jäckel, 2002). If we were to write the relevant expectation as an integral, we would find that its dimension is large or infinite. This is precisely the setting in which Monte Carlo methods become attractive (Glassermann, 2004).

Weber (2011) states that the Monte Carlo method is widely used in the financial markets as a valuation tool. It is used with path-dependent options and in models with more than one state variable. It is sometimes preferred to partial differential equation (PDE) or tree methods, even in situations where these methods could work well — simply because its generality and its robustness in contexts where a portfolio of options is being valued.

In this paper, we consider the Monte Carlo approach to value a new range of listed exotic products called Can-Do options — this product range was launched by the Johannesburg Stock Exchange (JSE) on 8 January 2007. Can-Do options are similar to CBOE's Flex option range of products. It started out where clients wanted to have the ability to customise key contract terms like the expiry date. However, the suite of products quickly grew to include exotic options and structured products[1]. Kotzé & Oosthuizen (2013) discuss and explain the local volatility pricing of exotic Can-Do options like Barrier options, as well as the methodologies used to determine their initial margins. Local volatility models have been in use since the 1980s although these were not known by the name "local volatility." The mathematical framework for local volatility was first formulated by Dupire (1994). At the same time, Derman & Kani (1994) and Rubinstein (1994) solved this problem numerically by implementing binomial trees. These methods have subsequently been improved by many other researchers (Andersen & Andreasen, 2000; Lagnado & Osher, 1997). It has since been realised that Dupire's framework is an extension of research done by Gyöngy (1986).

Many exotic options, like Barrier options, have Black-Scholes type closed form valuation formulas (Rubinstein & Reiner, 1991; Haug, 2007; Bouzoubaa & Osserein, 2010). However, it is also known that these formulas do not lead to market related and realistic prices and hedge ratios. This is due to the fact that these formulas use a fixed volatility. However, such option are path-dependent meaning that the actual path the stock takes to get to the expiry value on the expiry date, actually matters. To price them correctly one should either use stochastic volatility models or local volatility models. The choice here is to use either finite difference techniques or Monte Carlo simulation. This note will focus on Monte Carlo techniques.

The layout of this paper is as follows: In section 2 we give some history on the origins of Monte Carlo simulation and section 3 gives a brief overview of exotic options. In section 4 we bring local volatility into the Black-Scholes framework and we discretise the Black-Scholes SDE. Section 5 is crucial where we show how to use Monte Carlo simulation when pricing options. Section 6 discusses Dupire's local volatility mapping and in section 7 we use Dupire and price a single barrier option. We conclude in section 8.

Note that there are a few Appendices where we elaborate on some of the theory described in this paper. Appendix A shows why Monte Carlo simulation can be used when pricing options and we show how to discretise the Black-Scholes stochastic differential equation. Appendices B and C discusses the important issues of pseudo-random numbers and the convergence of Monte Carlo simulation. Appendix D gives an overview of the deterministic volatility function used to generate the volatility surface for ALSI options. In Appendix E we discuss the closed-form pricing formulas

[1] `http://www.jse.co.za/Products/Equity-Derivatives-Market/Equity-Derivatives-Product-Detail/Can-Do_Futures_and_Options.aspx`

of single barrier options.

2. A bit of History

The ‘Monte Carlo’ method was developed by the physicists and mathematicians working on the Manhattan Project[2] during the second world war. The main character was Stanilaw Ulam. Ulam and Edward Teller developed the first thermonuclear weapon also known as the hydrogen bomb or H-bomb. Ulam was intensely interested in random processes. He relaxed by playing solitaire and poker. The name ‘Monte Carlo’ was coined by Nicholas Metropolis during 1947 because Ulam had often mentioned his uncle, Michal Ulam, “who just had to go to Monte Carlo” to gamble.

It all started in October 1943 when Ulam received an invitation to join the Manhattan Project at the secret Los Alamos Laboratory in New Mexico. His extensive mathematical background made him aware that statistical sampling techniques had fallen into desuetude because of the length and tediousness of the calculations. It is believed that the first real applicaion of the ‘statistical sampling method’ was undertaken by Enrico Fermi in the 1930s. Due to the computational issues, this method did not really take off.

The second world war brought much needed progress though. Both the British and Americans developed electronic computing machines. In the USA the ENIAC (Electronic Numerical Integrator And Computer) was developed at the University of Pennsylvania in Philadelphia during 1943. The primary function for which ENIAC was designed was the calculation of tables used in aiming artillery. In the United Kingdom, the “Colossus” computer was built on the theoretical framework set by Turing (1936). This machine was built at Bletchley Park in 1944 to enable the cracking of German Enigma codes. The ENIAC was somewhat similar to the earlier Colossus, but considerably larger and more flexible (Istrail & Marcus, 2013).

These earliest large-scale electronic digital computers, the British Colossus and the American ENIAC, did not store programs in memory. To set up these computers for a fresh task, it was necessary to modify some of the machine’s wiring, re-routing cables by hand and setting switches. The basic principle of the modern computer — the idea of controlling the machine’s operations by means of a program of coded instructions stored in the computer’s memory — was conceived by Alan Turing (Dyson, 2012).

But, Los Alamos had access to the ENIAC. Access to this toy convinced Ulam that Fermi’s statistical techniques should be resuscitated, and he discussed this idea with John von Neumann — a principle member of the Manhattan Project[3]. This triggered the spark that led to the Monte Carlo method. One of the first problems solved on the ENIAC in 1946 was a computational model of a thermonuclear reaction[4]. Metropolis

[2] `http://en.wikipedia.org/wiki/Manhattan_Project`

[3] `http://library.lanl.gov/cgi-bin/getfile?00326866.pdf`

[4] A thermonuclear reaction or nuclear fusion is the fusion of two light atomic nuclei into a single heavier nucleus by a collision of the two interacting particles at extremely high temperatures, with the consequent release of a relatively large amount of energy. This reaction is responsible for the energy produced in the sun.

& Ulam (1949) published the first unclassified paper on the Monte Carlo method in 1949.

Los Alamos got its own computer early in 1952. It was called the MANIAC (Mathematical Analyzer, Numerical Integrator, and Computer or Mathematical Analyzer, Numerator, Integrator, and Computer). Enrico Fermi joined Los Alamos during the summer of 1952 and used to MANIAC to solve many statistical problems. A significant advance in the use of the Monte Carlo method came out of Nicholas Metropolis collaboration with Edward Teller. Together they introduced the idea of what is today known as importance sampling, also referred to as the Metropolis algorithm[5].

3. Exotic Options

> Can-Do Options and Futures are derivative products that give investors the advantages of listed derivatives with the flexibility of "over the counter" (OTC) contracts. Investors can negotiate the terms of an option's contract, choosing the type of option, underlying asset as well as the expiry date. Futures on bespoke baskets of shares and exotic options are very popular.

Two questions come to mind, "what is an exotic option" and, "what is a structured product?" Simply put, an exotic option is any type of option other than the standard calls and puts found on major exchanges. We can narrow this definition down slightly, by stating that exotic options are options for which payoffs at maturity cannot be replicated by a set of standard options (de Weert, 2008). Further to this, a structured derivative product is a bespoke instrument that enables an investor to pursue strategies tailored to his or her market view (Tan, 2010). Such a product allows an investor more control over the yield-risk tradeoff in his investment.

Exotic options and structured notes have traditionally been traded over-the-counter (OTC). The JSE was the first exchange in the world to list such products. Since 2007, the types of exotic listed on the JSE have grown tremendously. This forced Safex to divide exotic Can-Do options into two categories: vanilla exotics and complex exotics. Vanilla exotics include the more "standard" exotics, like fixed and floating strike lookbacks, Barriers, Asians and Binaries - we can throw variance futures into this mold as well . A complex exotic might be an option where one has a lookback option, but it has a barrier or Asian feature embedded, or it is a type of spread option on a basket of shares (meaning correlations play a role).

All JSE listed exotic options are European in nature — this means they can only be exercised on the expiry date. Most equity exotics have the FTSE/JSE Top 40 Index (ALSI) and FTSE/JSE Shareholders Weighted Top 40 Index (DTOP) indices as underlying instruments. On the foreign exchange side, the USDZAR is the preferred underlyer due to its massive liquidity.

If an instrument is liquid, a full MtM process can be run because on-screen traded prices or bid-ask spreads are available, and can be used at end of each day. However,

[5] `http://library.lanl.gov/cgi-bin/getfile?00326886.pdf`

all exotic Can-Do instruments are very illiquid, and a mark-to-model process is used. This means models are used in estimating the end of day levels. In this note, we will describe how these exotic instruments can be evaluated using Monte Carlo simulation.

4. Solving the Generalised Black-Scholes PDE

This section discusses the mathematical background and can be skipped

Option pricing theory due to Fischer Black, Myron Scholes and Robert Merton is now well established and understood (Kotzé, 2003). Stewart (2012) stated that the Black-Scholes model is one of his 17 equations that changed the world. He calls it the "Midas" equation.

In the so-called Black-Scholes world we make, inter alia, the following assumptions

- Stock prices follow a continuous random walk Brownian motion
- The efficient market hypothesis holds
- Investors live in a risk-neutral world
- Delta-hedging is done continuously.

In general, Black & Scholes assumed that the financial market is a system that is in equilibrium — without outside or exogenous influences, the system is at rest; everything balances out and supply equals demand. Any distortion or perturbation is thus quickly handled by the market players and equilibrium restored

The Brownian motion of asset prices through time is shown in Figure 1.

4.1. The Black-Scholes PDE under Local Volatility

The seminal Black-Scholes-Merton option pricing formula can be obtained analytically under the assumptions of a constant volatility, constant risk-free rate and constant dividend yield. As a matter of fact, it can be derived in four ways[6]:

- Straightforward integration
- Applying the Feynman-Kac theorem
- By transforming the Black Scholes partial differential equation (PDE) into the heat equation. This is a well-known parabolic PDE formulated by Joseph Fourier in 1822 (Narasihan, 1999, 2009). This is the original approach adopted by Black and Scholes
- Using the capital asset pricing model (CAPM).

[6] `http://www.frouah.com/financenotes/BlackScholesFormula.pdf`

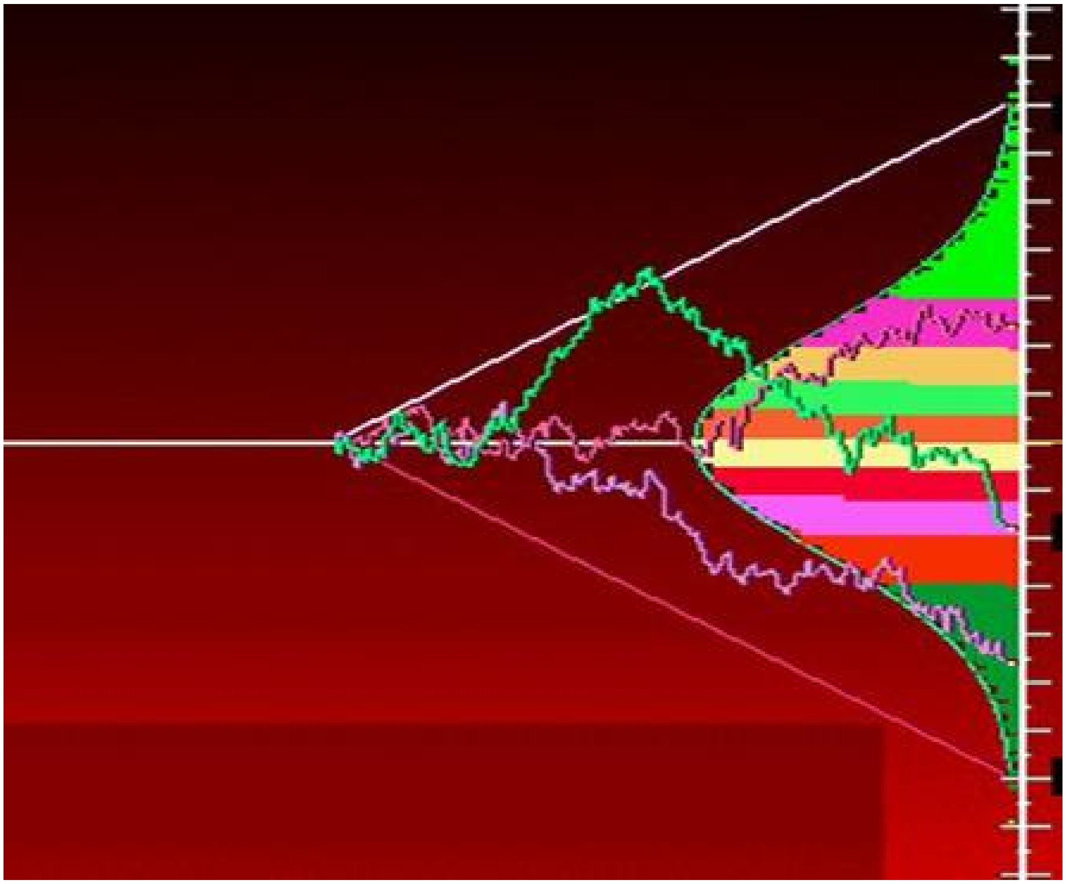

Figure 1: Share prices diffuse through time governed by a standardised normal distribution.

It can also be shown that the theory is consistent with an implied volatility skew (Hull, 2012).

Although satisfactory for European options, the Black-Scholes model comes up short for more complex options, such as Asian options (whose payoff depends on the average price of the underlying asset over time), barrier options (whose value depends on whether a specific boundary value has been attained by the underlying asset before its maturity) or even common American options.

The theory thus needed to be extended. If we now generalise the standard Black-Scholes stochastic differential equation (SDE) and assume that volatility is dependent on the asset's price and time (it's not constant anymore) but we still assume it to be deterministic, we get

$$dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t. \tag{4.1}$$

Remember, W_t is a standard Brownian motion and as such $dW_t = \varepsilon\sqrt{dt}$ where $\varepsilon \sim N(0,1)$, $N(0,1)$ being a standardised normal distribution.

In Equation (4.1), the function $\sigma(S,t)$ is called the local volatility function because it is dependent on both S and t. Note that $\sigma(t)$ is sometimes referred to as the instantaneous volatility — it is a function of time only. See Kotzé et al. (2014) for a full description and explanation of the concept of local volatility. The local volatility is the instantaneous volatility for each point in space and time i.e., it is the volatility that holds near the point when the stock's value is S_t at a time t. It is the volatility that is 'local' to the point (S_t, t) — 'local' defined in a similar fashion to the 'local' in 'local extrema'. Further to this definition, this description is similar to the definition of a 'field' in physics. In physics, a 'field' is a physical quantity that has a value for each point in space and time. In this case, local volatility is a scaler field (Boas, 1983; Reif, 2008). These concepts come from *mean field theory* (MFT) where the *Ising* model is a standard many-body system discussed in solid state physics textbooks (Harras, 2012; McCauley, 2013; Sornette, 2014).

Please note that the basic Black-Scholes assumptions still hold: the asset price S_t evolves log-normally, μ is the expected continuously compounded rate of return earned by an investor in a short period of time dt — the instantaneous expected return and W_t is a standard Brownian motion or Wiener process. It is clear that W, and consequently its infinitesimal increment dW_t, still represents the only source of uncertainty in the price history of the security.

Black, Scholes and Merton made some assumptions in order to facilitate a better understanding of the dynamics of the security price S_t. One of the main assumptions is that of risk neutrality. In its simplest form, this infers that all risk-free portfolios can be assumed to earn the same risk-free rate. We can then put $\mu = r_t - d_t$ where r_t is a deterministic interest rate (it can be obtained from a relevant yield curve) and d_t is a deterministic dividend yield. Under these assumptions, the risk-neutral dynamic of the asset is (Hull, 2012)

$$dS_t = (r_t - d_t)S_t dt + \sigma(S_t, t)S_t dW_t. \tag{4.2}$$

To move forward and obtain the price of an option, we let a scalar function $V_l(S, t)$ be the value of a contingent claim like an option at any time t conditional on the price of the underlying being S_t at that time. Using Ito's lemma, equation (4.2) can be transformed to the generalised Black-Scholes stochastic partial differential equation (PDE)

$$\frac{\partial V_l}{\partial t} + \frac{1}{2}\sigma^2(S_t, t)S_t^2 \frac{\partial^2 V_l}{\partial S_t^2} + (r_t - d_t)S\frac{\partial V_l}{\partial S_t} - r_t V_l = 0. \tag{4.3}$$

Equation (4.3) basically describes how the value of a derivative contract, at a continuum of potential future scenarios, diffuses *backwards* in time towards today. This equation is a backward parabolic partial differential equation also known as the *backward Kolmogorov* equation (Rebonato, 2004; Duffie, 1996). This is just a extention of Joseph Fourier's one-dimensional heat conduction equation formulated in 1822 (Narasihan, 1999).

4.2. Discretising the Black-Scholes Equation

Fourier solved his simplistic heat conduction equation analytically by introducing Fourier transforms. The extended version is not solved that easily. However, we will understand the SDE in Equation (4.2) much better if we make a change of variables. Let's re-write (4.2) in terms of $\ln(S)$ and then a simple application of Itô's lemma gives

$$S_T = S_0 \exp\left(\left((r_T - d_T) - \frac{1}{2}\sigma^2(S_T, T)\right) T + \sigma(S_T, T)\varepsilon\sqrt{(T)}\right). \tag{4.4}$$

Note, $\varepsilon \sim N(0,1)$, $N(0,1)$ being a standardised normal distribution. See Appendix A for the derivation. Equation (4.4) formulates a way to obtain the terminal value of the stochastic process S. This, together with equations (5.8) (5.9) (see section 5 below) can now be used to obtain the value of our option $V(S,t)$.

Equations (4.2) and (4.4) are both defined for a continuous time variable t. So the question is how do we sample from the continuous distribution for the variable S_T? These equations can be discretised by using the Euler scheme. This leads to

$$S(t_{i+1}) = S(t_i) \exp\left[\left((r(t_i) - d(t_i)) - \frac{\sigma^2(S(t_i), t_i)}{2}\right) \Delta t + \sigma(S(t_i), t_i)\, \varepsilon_t \sqrt{\Delta t}\right]. \tag{4.5}$$

Here, $i = 1, 2, \ldots, N$ such that $t_i = i\Delta t$ and $T = N\Delta t$. In order to start the simulation we need a starting asset value $S(t_0)$. If we then have the input parameters like the volatilities, risk-free rates and dividend yields, we can estimate a price for S at each discretised step i until we reach $S(t_N) = S(T)$. Such a price path is shown in Figure 2 where we have 25 time steps.

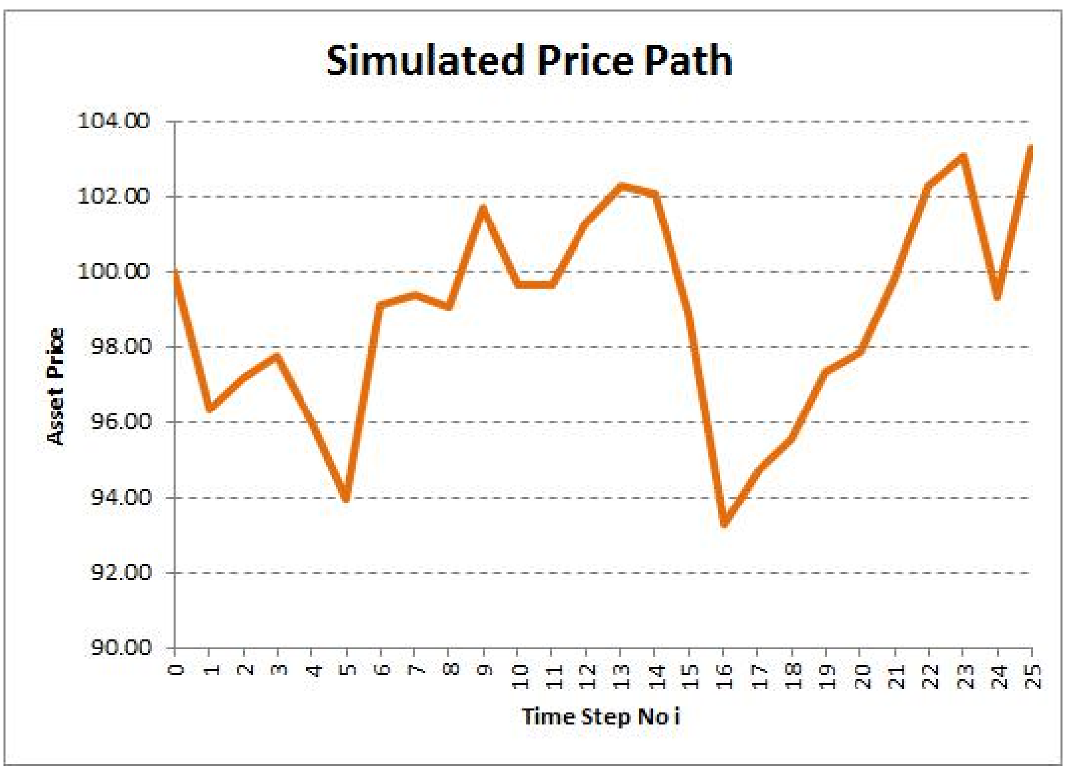

Figure 2: A price path for a security with price R100 at time $t = t_0$, risk-free rate $r = 0.05$, dividend yield $d = 0.025$ (both continuous), volatility of 15% and $T = 0.5$. Further, $N = 25$ and then $\Delta t = 0.02$

The Euler scheme can be improved if we include the next order terms of the Itô-

Taylor expansion of Equation (4.1). This gives (Jäckel, 2002; Glassermann, 2004; Clark, 2011)

$$S(t_{i+1}) = S(t_i)\exp\left[\left((r(t_i) - d(t_i)) - \frac{\sigma^2(S(t_i),t_i)}{2}\left[\varepsilon_t^2 - 1\right]\right)\Delta t + \sigma(S(t_i),t_i)\,\varepsilon_t\sqrt{\Delta t}\right]. \tag{4.6}$$

ε_t is sampled from a standardised normal distribution — this is further discussed in §6. By adding a term where the diffusion is $O(\Delta t)$ we get convergence of strong order 1. One of the advantages of Milstein over Euler time stepping is improved convergence when Δt is infinitesimal. In that case we can take larger time steps and get by with a smaller number of time steps N.

5. From Black-Scholes to Monte Carlo Simulation

Let's assume $V_l(S_T, T)$ is the final condition of our contingent claim at expiry T and, given that the process, S, starts at S_0 at initial time t_0. The general solution to the Black-Scholes backward parabolic partial differential equation in Equation (4.3) is given by the Feyman-Kac theorem stating

$$V_l(S_0, t_0) = E^{\mathbb{Q}}\left[e^{-\int_{t_0}^{T} r_u du} V_l(S_T, T) | S_{t_0} = S_0\right], \tag{5.7}$$

where $S,\ t \in \mathbb{R}_0^+$ and S_t is described by the stochastic differential Equation (4.2) and r_u is the instantaneous discount rate applicable for a very short period of time du (Linetsky, 1998; Duffie, 1996). Note that the expectation is taken under the risk-neutral probability measure $\mathbb{Q}$ where the stochastic term in Equation (4.2) is governed by Brownian motion or it is a Wiener process. Note that the Feyman-Kac theorem provides the justification for the practice of evaluating today's value of an option $(V_l(S_0, t_0))$ as the discounted expectation of its terminal payoff.

Using the mathematical law of expectation, the expectation for a call option in Equation (5.7) can be written as an integral such that (Duffie, 1996; Wilmott, 2000)

$$V(S,t) = e^{-r_T(T-t)} \int_K^\infty \max[0, (S_T - K)] g(S_T) dS_T \tag{5.8}$$

where $g(S_T)$ is the probability density function (pdf) of S_T and we assume $\ln(S_T)$ is normally distributed with a standard deviation of w. We thus need to integrate over all possible S-values that is larger than the strike K at expiry. For a put we have

$$V(S,t) = e^{-r_T(T-t)} \int_0^K \max[0, (K - S_T)] g(S_T) dS_T. \tag{5.9}$$

Remember $K, S \in \mathbb{R}_0^+$

Using equations (5.8) and (5.9), we show in Appendix A.5 that these equations can be discretised such that the simplest Monte Carlo method to price an option is given by

$$V_{MC}(S,t) = e^{-r_T(T-t)}\frac{1}{M}\sum_{i=1}^{M}\max[0,\phi(S_T-K)] \tag{5.10}$$

where S_T is attained after N time steps that coincide with the expiry time T. We can use either Equation (4.5) or Equation (4.6) to estimate S_T. To obtain the Monte Carlo option price, we need to obtain M, S_T values. This means we simulate S_T, M times to obtain the average option value V_{MC}. Note: N is the number of time steps and M the number of simulations.

By scrutinising equations (5.10), (4.5) and (4.6) it becomes clear that MC methods are indeed techniques utilising random numbers and probability to solve problems. It is evident that such an analysis is based on artificially recreating a chance process, running it many times and directly observing the results.

Figure 3 shows 5 price paths generated with Equation (4.5), each having 25 time steps. Here we have a fixed volatility, interest rate and dividend yield (in the limit as $M \to \infty$, all S_T's will have a normal distribution as shown in Figure 1). If we have a call option with a strike price of 100, Equation (5.10) leads to an option value of R11.02. This is shown in Table 1.

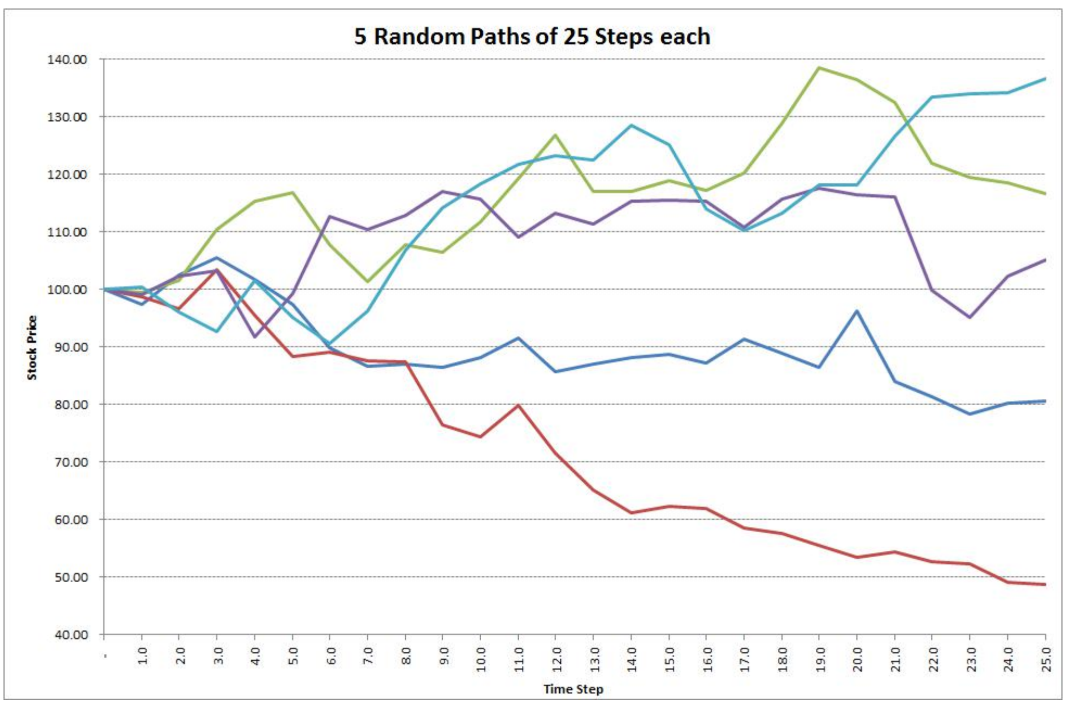

Figure 3: Price paths for a security with price R100 at time $t = t_0$, risk-free rate $r = 0.05$, dividend yield $d = 0.025$ (both continuous), volatility of 0.25 and $T = 1.0$. Further, $N = 25$ and then $\Delta t = 0.04$ and $M = 5$

In the example above we used a fixed volatility of 25%. However, crucial to obtaining the correct terminal values S_T is that the volatilities we use in equations (4.5) and (4.6) are the volatilities obtained from a local volatility surface. We thus need to understand what we mean by the time stamp in the local volatility $\sigma(S(t_i), t_i)$

Time Steps	S **Path 1**	S **Path 2**	S **Path 3**	S **Path 4**	S **Path 5**
t=0	100.00	100.00	100.00	100.00	100.00
1.0	97.49	98.66	99.55	99.19	100.42
2.0	102.50	96.74	101.50	102.40	96.10
3.0	105.45	103.45	110.48	103.19	92.63
4.0	101.84	95.56	115.39	91.72	101.54
5.0	97.43	88.44	116.87	99.39	95.19
6.0	89.92	89.17	107.82	112.78	90.70
7.0	86.61	87.65	101.33	110.39	96.34
8.0	87.03	87.35	107.89	112.91	106.92
9.0	86.52	76.50	106.55	117.03	114.14
10.0	88.12	74.35	111.81	115.73	118.42
11.0	91.51	79.93	119.28	109.11	121.68
12.0	85.74	71.59	126.84	113.27	123.20
13.0	87.00	65.09	117.05	111.47	122.55
14.0	88.18	61.22	117.10	115.39	128.60
15.0	88.78	62.40	118.89	115.56	125.20
16.0	87.23	61.89	117.24	115.35	113.94
17.0	91.47	58.64	120.21	110.88	110.23
18.0	88.94	57.62	128.98	115.69	113.36
19.0	86.45	55.57	138.58	117.54	118.14
20.0	96.29	53.43	136.41	116.45	118.08
21.0	84.00	54.48	132.53	116.12	126.71
22.0	81.40	52.66	121.86	99.82	133.38
23.0	78.38	52.31	119.51	95.14	134.03
24.0	80.22	49.19	118.52	102.38	134.16
25.0 (S_T)	80.68	48.68	116.72	105.11	136.68
Call Value at T	**0.00**	**0.00**	**16.72**	**5.11**	**36.68**
Average Value	**11.70**				
Value Today	**11.02**				

Table 1: Five price paths and the Monte Carlo option price for a vanilla call. The parameters are given below Figure 3

in these equations. This shows we first of all need the stock price at each time step, i.e., $S(t_i)$. We have given some examples in Table 1. But, further to this, we also need the instantaneous volatility at each time step for each stock price. We can obtain all of this from a three dimensional local volatility surface. We will discuss this in section 8 below.

6. Random Number Generators (RNG)

Monte Carlo simulation is done by implementing Equation (5.10). However, we need S_T and we use equations (4.5) and (4.6) for that purpose. From these equations it is evident that we need a random number ε that is one of the inputs. $\varepsilon \sim N(0,1)$ and is drawn from a standardised normal distribution. In practice the random number is sampled from a discrete distribution calculated by a computer. As such we call these random numbers pseudorandom numbers because they are generated by a computer algorithm utilising mathematical formulae. They are not true random numbers. True randomness can only be obtained from natural phenomena like radiocative decay or atmospheric noise.

Many pseudorandom number generators have been developed over the past few decades. One of the generators used by many practitioners is the Mersenne Twister[7] (Jäckel, 2002). This algorithm has been implemented in many programming languages like C++ and even VBA[8]. Another excellent RNG is the Park-Miller algorithm with Bays-Durham shuffle (Park & Miller, 1988).

Most RNG generate uniform random numbers. This means these numbers are drawn from a uniform distribution. However, we need normal random numbers. The Box-Muller transform is widely used (Jäckel, 2002; Glassermann, 2004). The JSE uses Box-Muller and both the Mersenne twister and the Park-Miller algorithm dependent on the implementation.

7. Monte Carlo Simulation and Convergence

In general, Monte Carlo methods give us at best a statistical error estimate. A Monte Carlo calculation usually follows the following steps: carry out the same procedure many times, take into account all of the individual results, and summarise them into an overall approximation to the problem in question. The approximation is usually the average. The numerically exact solution will be approached only as we iterate the procedure more and more times, eventually converging at infinity (Jäckel, 2002; Glassermann, 2004).

This will be very time consuming so we are not just interested in a method to converge to the correct answer after an infinite amount of calculation time, but rather we wish to have a good approximation quickly. Therefore, once we are satisfied that a particular Monte Carlo method works in the limit, we are naturally interested in its convergence behaviour, or, more specifically, its convergence speed.

[7] `http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html`

[8] `http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/VERSIONS/BASIC/basic.html`

Techniques have been developed to reduce the variance of the result and thus to reduce the number of simulations required for a given accuracy. Such techniques are called "variance reduction techniques."

The JSE mostly uses two techniques:

- Antithetic sampling;
- Control variates.

(Jäckel, 2002) and (Glassermann, 2004) give very good overviews of these techniques.

8. Local volatility

Local volatility models are widely used in the finance industry (Engelmann et al., 2009). Whereas stochastic volatility and jump-diffusion models introduce new risks into the modeling process, local volatility models stay close to the Black-Scholes theoretical framework and only introduce more flexibility to the volatility. This is one of the main reasons of fierce criticism of local volatility models (Ayache et al., 2004). Thus, it is a mistake to interpret local volatility as a complete representation of the true stochastic process driving the underlying asset price. Local volatility is merely a simplification that is practically useful for describing a price process with non-constant volatility. A local volatility model is a special case of the more general stochastic volatility models. That is why these models are also known as "*restricted stochastic volatility models*".

8.1. Dupire's Formula

The local volatility function $\sigma(S,t)$ is assumed to be deterministic — it is a deterministic function of a stochastic quantity S_t and time. So there is still just one source of randomness, ensuring the completeness of the Black-Scholes model is preserved. Completeness is important, because it guarantees unique prices, thus arbitrage pricing and hedging (Dupire, 1993).

Dupire (1994) was the first to show algebraically that, given prices of European call or put options across all strikes and maturities, we may deduce the volatility function $\sigma(S,t)$, which produces those prices via the full Black-Scholes equation (Clark, 2011). Dupire's insight was that if the spot price follows a risk-neutral random walk and if no-arbitrage market prices for European vanilla options are available for all strikes and expiries, then the local volatility $\sigma(S,t)$ in Equation (4.1) can be extracted analytically from European option prices (Dupire, 1993). He, unknowingly, applied Gyöngy's theorem (Gyöngy, 1986).

Dupire showed that if we have implied or market volatilities, we can calculate the local volatilities thereof where (Wilmott (1998) and Clark (2011))

$$\sigma^2_{loc}(S_0,K,\tau) = \frac{\sigma^2_{imp} + 2\tau\sigma_{imp}\dfrac{\partial\sigma_{imp}}{\partial\tau} + 2(r-d)K\tau\sigma_{imp}\dfrac{\partial\sigma_{imp}}{\partial K}}{\left(1 + Kd_1\sqrt{\tau}\dfrac{\partial\sigma_{imp}}{\partial K}\right)^2 + K^2\tau\sigma_{imp}\left(\dfrac{\partial^2\sigma_{imp}}{\partial K^2} - d_1\sqrt{\tau}\left(\dfrac{\partial\sigma_{imp}}{\partial K}\right)^2\right)}, \tag{8.11}$$

where

$$d_1 = \frac{\ln(S_0/K) + \left((r-d) + \sigma_{imp}^2/2\right)\tau}{\sigma_{imp}\sqrt{\tau}},$$

and $\tau = T - t$ such that t and S_0 are respectively the market date, on which the volatility smile is observed, and the asset price on that date. Note that Equation (8.11) gives the variance, i.e., σ^2. See Kotzé et al. (2014) for the derivation.

The main problem is that the implied or traded volatilities are only known at discrete strikes K and expiries T. This is why the parameterisation chosen for the implied volatility surface is very important. If implied volatilities are used directly from the market, the derivatives in Equation (8.11) needs to be obtained numerically using finite difference or other well-known techniques. This can still lead to an unstable local volatility surface. Furthermore we will have to interpolate and extrapolate the given data points unto a surface. Since obtaining the local volatility from the data involves taking derivatives, the extrapolated implied volatility surface cannot be too uneven. If it is, this unevenness will be exacerbated in the local volatility surface showing that it is not arbitrage free in these areas.

Kotzé et al. (2014) showed that the JSE uses a functional form for their ALSI implied volatility surface. This function is quadratic across strike and exponential across time. This three dimensional function is fitted to traded data. They further showed that all derivatives in Equation (8.11) can then be obtained analytically and the ALSI local volatility surface is easy to calculate and obtain. They went further and discussed the DTOP and USDZAR implied volatility surfaces. There are no functional forms available and all derivatives in Equation (8.11) needs to be computed numerically. We expand on this in the next section.

> In the foreign exchange market, options are traded on the Delta — effectively a measure of the moneyness — as opposed to the absolute level of the strike. See Clark (2011) for the FX version of Equation (8.11)

8.2. Dupire and Monte Carlo Simulation

The JSE uses Dupire's formula in Equation (8.11) to convert the implied volatility surfaces for all vanilla options traded on all underlying future contracts to their respective local volatility surfaces. The local volatility surfaces are used when exotic options are evaluated. Exotics are mostly traded on the ALSI, DTOP and USDZAR and some single name futures.

Figures 4 and 5 show the implied and local volatility surfaces for ALSI and USDZAR options respectively on 28 May 2014.

From Figure 4 we notice that the implied volatility surface does not have a lot of curvature — it is skewed but flat. However, we also see from the local volatility surface that it has more curvature. This shows that the local volatility skew is twice that of the implied volatility skew. Figure 5, shows the USDZAR implied volatility surface that has the currency market's all familiar smile. Here we also show the local volatility surface with steeper sides.

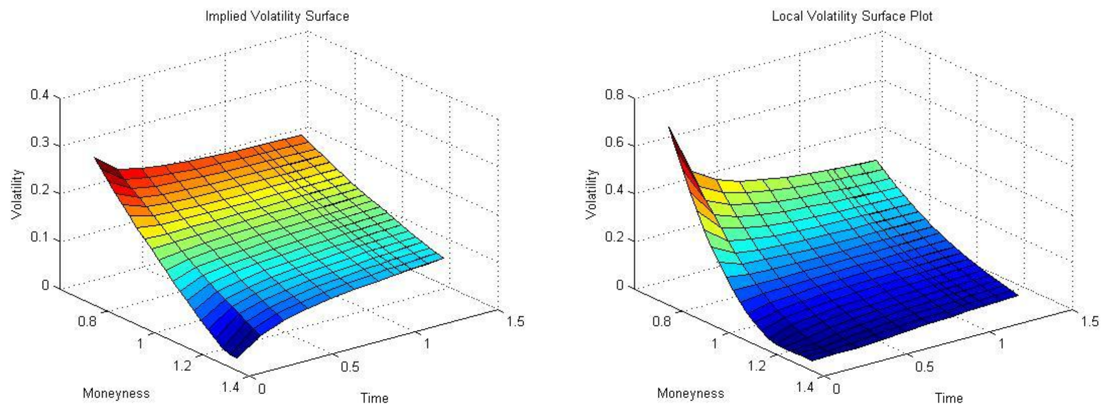

Figure 4: ALSI implied and local volatility surfaces on 28 May 2014

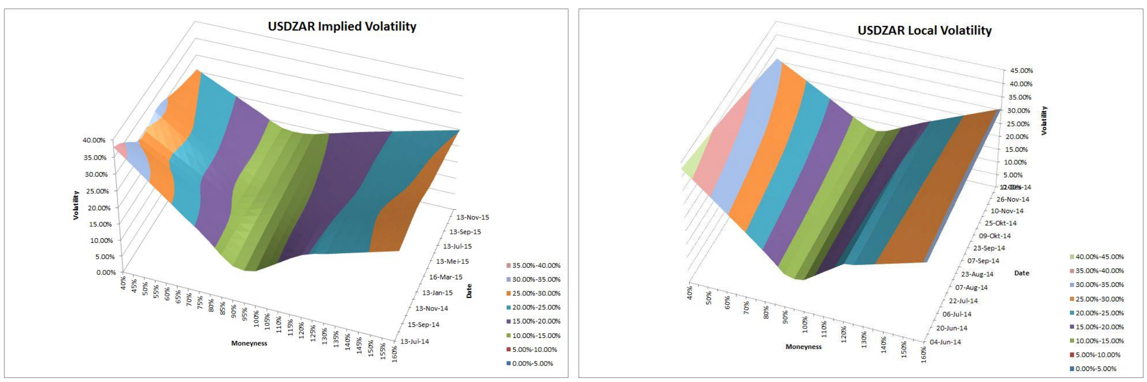

Figure 5: USDZAR implied and local volatility surfaces on 28 May 2014

The JSE generates the ALSI implied volatility surface by fitting a three dimensional deterministic surface function to traded data. We summarise the approach in Appendix B. This means we have a smooth arbitrage-free surface. On the other hand, the implied volatility surfaces for the DTOP and USDZAR are given in discrete form only — these are not smooth.

Continuing with our example: in section 5 and Table 1 we tabulated some price paths. We now want to calculate the Dupire local volatility for each stock price at each time step. This is the local volatility that should then be used in equations (4.5) and (4.6) to obtain the price paths as shown in Table 1.

On a practical note: in equations (4.5) and (4.6) we generate a stock price $S(t_i)$ at each time step t_i. To apply Equation (8.11) we now say that $\tau = t_i$ and $S(t_i) = K$. Why? To obtain the local volatility we step forward in time and at every time step assume we price an option with an expiry time of T and then $\tau = T - t_0$ but in most cases $t_0 = 0$. Further, Dupire's equation is given in terms of the strike. It holds for all strikes because $K \in \mathbb{R}_0^+$. We then say that $S(t_i)$ is a possible strike at time t_i and we have $K = S(t_i)$.

In our example, the price paths were shown for a one year time period. The

JSE/FTSE Top 40 index was 44,732 on 28 May 2014. The price paths in Table 1 were generated with a fixed volatility of 25%. We thus cannot generate the same price paths under a local volatility regime. However, to show the difference between a fixed volatility and local volatility implementation, we now use the same random numbers as before and we assume that the one year ATM volatility is 25%. So we run this experiment and generate 5 price paths under the ALSI local volatility surface. The implied volatility surface is underpinned by the parameters shown in Tables 5 and 6 and the local volatility surface generated by Equation (8.11). The newly generated price paths are shown in Figure 6. The actual numbers are listed in Table 2 and the corresponding local volatilities are listed in Table 3.

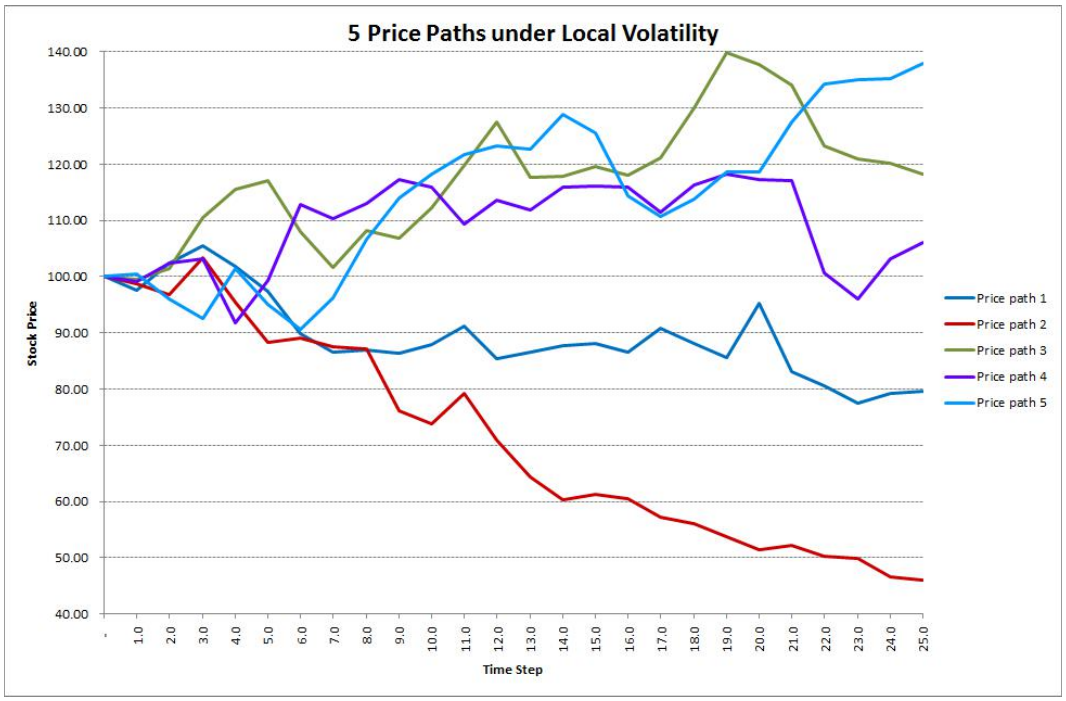

Figure 6: Price paths for a security with price R100 at time $t = t_0$, risk-free rate $r = 0.05$, dividend yield $d = 0.025$ (both continuous) and $T = 1.0$. Further, $N = 25$ and then $\Delta t = 0.04$ and $M = 5$. The volatility used is the local volatility for ALSI options on 28 May 2014.

Comparing graphs 6 and 3 and Tables 2 and 1 reveal that the stock prices are not that much different. This is the way it should be because the local volatility does not differ that much from the implied volatility. However, even these slight difference, can lead to vastly different exotic option prices and especially, Greeks.

9. Pricing Barrier Options under Local Volatility

Let's now look at the price and hedge ratio Delta of a down-and-out put barrier option on the JSE/FTSE Top 40 index. We price this option using Monte Carlo simulation under a local volatility surface and using the closed-form solutions. Rubinstein & Reiner (1991) derived closed-form solutions to all vanilla barrier options in a

Time Steps	**Path 1**	**Path 2**	**Path 3**	**Path 4**	**Path 5**
0	100.00	100.00	100.00	100.00	100.00
1.0	97.49	98.66	99.55	99.19	100.42
2.0	102.47	96.72	101.50	102.40	96.11
3.0	105.44	103.41	110.49	103.20	92.60
4.0	101.87	95.54	115.46	91.75	101.45
5.0	97.47	88.40	117.03	99.35	95.11
6.0	89.94	89.05	108.05	112.74	90.60
7.0	86.57	87.46	101.58	110.41	96.17
8.0	86.91	87.08	108.17	112.99	106.70
9.0	86.32	76.20	106.87	117.18	113.95
10.0	87.83	73.91	112.18	115.95	118.29
11.0	91.15	79.29	119.74	109.38	121.63
12.0	85.36	70.91	127.41	113.60	123.24
13.0	86.54	64.32	117.68	111.86	122.68
14.0	87.64	60.31	117.80	115.84	128.82
15.0	88.17	61.26	119.67	116.08	125.53
16.0	86.58	60.55	118.08	115.94	114.31
17.0	90.73	57.18	121.15	111.50	110.65
18.0	88.18	55.97	130.07	116.40	113.84
19.0	85.67	53.78	139.87	118.31	118.70
20.0	95.34	51.50	137.80	117.30	118.71
21.0	83.16	52.30	133.99	117.03	127.47
22.0	80.51	50.34	123.31	100.65	134.27
23.0	77.44	49.80	121.01	95.94	135.04
24.0	79.16	46.63	120.08	103.23	135.28
25.0	79.53	45.94	118.33	106.00	137.92

Table 2: Price paths under a local volatility regime

$\tau = t_i$	**LV Path 1**	**LV Path 2**	**LV Path 3**	**LV Path 4**	**LV Path 5**
0	24.9996%	24.9996%	24.9996%	24.9996%	24.9996%
0.04	27.4866%	26.2902%	25.4045%	25.7573%	24.5469%
0.08	23.0088%	27.6461%	23.7543%	23.0681%	28.1790%
0.12	21.2633%	22.5733%	18.2763%	22.7096%	30.5823%
0.16	23.6806%	27.9284%	16.3403%	30.7435%	23.9482%
0.20	26.4488%	32.8200%	16.1544%	25.2543%	28.0043%
0.24	31.2864%	31.9153%	20.5232%	18.3454%	30.8294%
0.28	33.3024%	32.6808%	23.9452%	19.6009%	27.0274%
0.32	32.7299%	32.6118%	20.7567%	18.6921%	21.4314%
0.36	32.8310%	39.9575%	21.4424%	17.2989%	18.4947%
0.40	31.6159%	41.1589%	19.3257%	17.9228%	17.1125%
0.44	29.4515%	36.9318%	16.8225%	20.5418%	16.2359%
0.48	32.7295%	42.4271%	14.8280%	19.0348%	15.9505%
0.52	31.8303%	46.6896%	17.7768%	19.7628%	16.2770%
0.56	31.0305%	49.0341%	17.8591%	18.4809%	14.8622%
0.60	30.5979%	47.9364%	17.4077%	18.5037%	15.8216%
0.64	31.3536%	48.0069%	17.9798%	18.6365%	19.1543%
0.68	29.0183%	49.8401%	17.2056%	20.1550%	20.4486%
0.72	30.2556%	50.2336%	15.1812%	18.6495%	19.4400%
0.76	31.4912%	51.2306%	13.5589%	18.1646%	18.0554%
0.80	26.5871%	52.2391%	14.0215%	18.5222%	18.1259%
0.84	32.6223%	51.4452%	14.7868%	18.6588%	16.0669%
0.88	33.9686%	52.2491%	17.0924%	24.2576%	14.8494%
0.92	35.5782%	52.2521%	17.7223%	26.1977%	14.8259%
0.96	34.4757%	53.6275%	18.0146%	23.2429%	14.8868%
1.00	34.1544%	53.6906%	18.5148%	22.2471%	14.5786%

Table 3: The Dupire local volatilities as obtained from the ALSI local volatility surface on 28 May 2014

Black-Scholes framework. We discuss these in Appendix C and the pricing equations are given in Equation (C.34).

To explain the differences between the MC and closed-form solutions, we look at an example of a one month down-and-out put. This example's input parameters are shown in Table 4.

Description	Input Values
Equity price	**44 732.00**
Strike	**44 732.00**
Barrier	**40 258.80**
Rebate	**0.000**
Number of discrete observations	**146.00**
Current date	**28-May-14**
Maturity date	**30-Jun-14**
Interest Rate (NACA)	**6.00%**
Volatility	**14.50%**
Dividend Yield (NACA)	**3.00%**
Type of option	**Down and out put**

Table 4: Input parameters for a down-and-out-put option on the JSE/FTSE Top 40 index.

The payoff function for a down-and-out barrier is given by

$$V_{DAO}(S,t) = e^{-r_T(T-t)}\frac{1}{M}\sum_{j=1}^{M}\begin{cases} \max\left[0, \phi\left(S_T - K\right)\right] & \iff \quad S(t_i) > H,\ t \leq T \\ R & \iff \quad S(t_i) \leq H,\ t \leq T \end{cases}$$

Monte Carlo simulation is implemented where we use Equation (5.10). We can further use either Equation (4.5) or Equation (4.6) to estimate the stock price $S(t_i)$ at each time step t_i and ultimately S_T. Having calculated the stock price at each time step t_i, makes it quite easy to implement the boundary conditions. At each time step one needs to check if the stock price $S(t_i)$ is above or below the barrier H.

If we want to implement the closed-form solution, we need to understand that we can do it in two different ways: we first price it using a fixed volatility of 14.5% and secondly we obtain the volatility from the implied volatility surface.

The price dynamics of this option is shown in Figure 7 where closed-form is abbreviated by CF. The barrier is 90% of the spot level and it is short dated. The price dynamics between the three methods do not differ much and using the slower Monte Carlo method does not add much value. Figure 7 shows the familiar option profile for a down-and-out put option — the option vanishes if the stock price breaches the barrier level.

The dynamics of the hedge ratio Delta is shown in Figure 8. Here is where the methods differ substantially. Far from the barrier all three methods give the same Delta. However, interestingly, close to the barrier, the closed-form solution breaks down. This is a put option and the Δ should be negative always. The local volatility

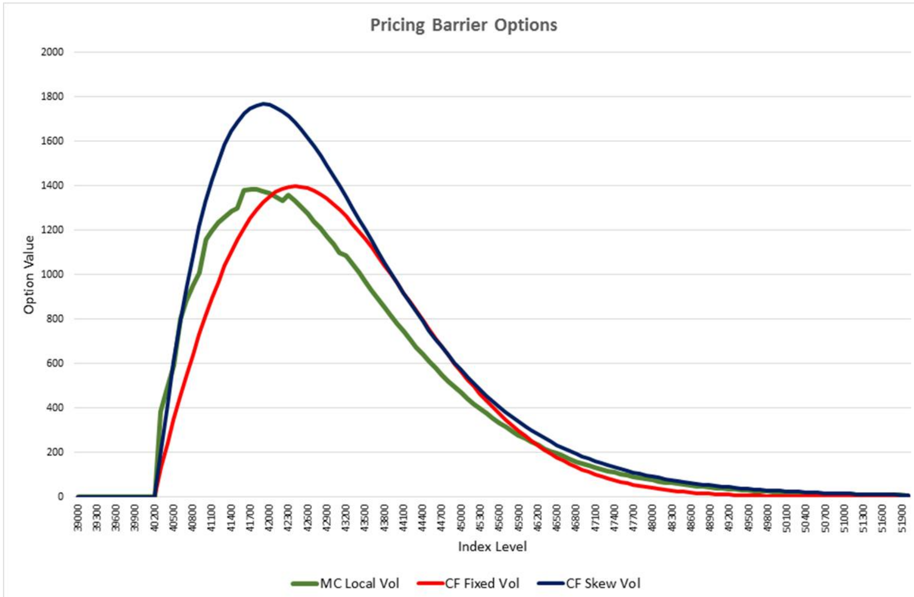

Figure 7: Price dynamics for a down-and-out-put.

model behaves correctly and gives the correct hedge parameter even if the spot is very close to the barrier.

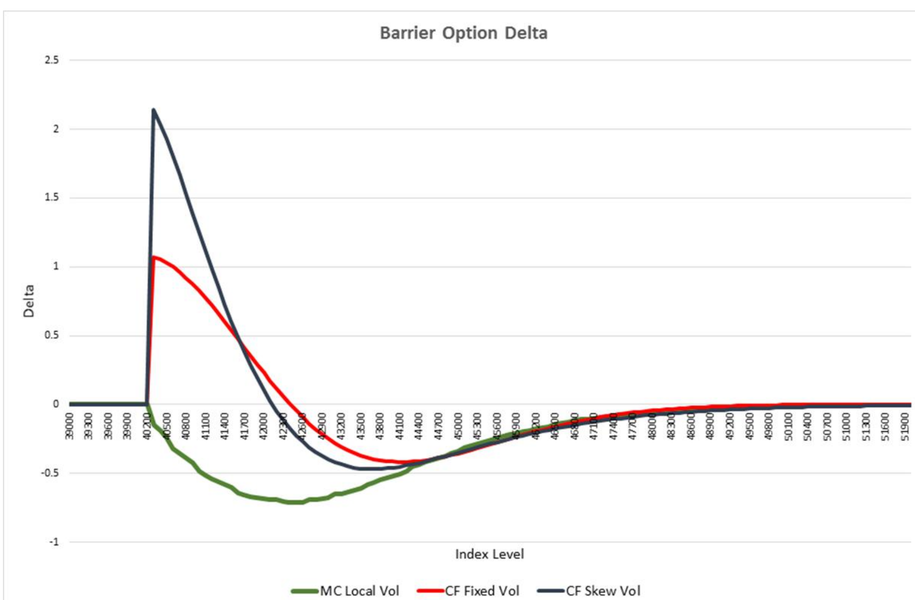

Figure 8: The Δ-dynamics for a down-and-out-put option.

10. Conclusion

Monte Carlo methods are powerful and can be used to price exotic options. In this note we introduced Monte Carlo simulation and explained why it can be used to price all kinds of derivatives securities. We introduced the local volatility framework and showed how to incorporate it into a MC simulation. This was done at the hand of many examples. We concluded by explaining how a barrier option should be priced in a local volatility world.

Acknowledgements

Thanks are due to the JSE for data.

Appendices

A. From Black-Scholes to Discrete Monte Carlo Simulation

A.1. The Feynman-Kac Theorem and Expectation

Fourier solved his simplistic heat conduction equation analytically by introducing Fourier transforms. The extended version is not solved that easily. However, the Feynman-Kac theorem can be used to solve it (Rebonato, 2004). This is possible if $V_l(S,t)$ in Equation (4.3) is twice differentiable and $V_l(S_T,T)$ is the terminal condition. We also have S_T being the terminal asset value on the expiry time T. The Feynman-Kac theorem establishes a link between parabolic partial differential equations and stochastic processes or diffusion problems we encounter in finance (Jäckel, 2002). It offers a method of solving certain PDEs by simulating random paths of a stochastic process (Klebaner, 2005; Clark, 2011). If we now let $V_l(S_T,T)$ be the final condition of our contingent claim at expiry T and, given that the process, S, starts at S_0 at initial time t_0, the general solution to this backward parabolic partial differential equation shown in Equation (4.3) is given by

$$V_l(S_0,t_0) = E^{\mathbb{Q}}\left[e^{-\int_{t_0}^{T} r_u du} V_l(S_T,T) | S_{t_0} = S_0\right], \tag{A.12}$$

where $S,\ t \in \mathbb{R}_0^+$ and S_t is described by the stochastic differential Equation (4.2) and r_u is the instantaneous discount rate applicable for a very short period of time du (Linetsky, 1998; Duffie, 1996). Note that the expectation is taken under the risk-neutral probability measure $\mathbb{Q}$ where the stochastic term in Equation (4.2) is governed by Brownian motion or it is a Wiener process. Note that the Feyman-Kac theorem provides the justification for the practice of evaluating today's value of an option $(V_l(S_0,t_0))$ as the discounted expectation of its terminal payoff.

In general, if we assume the volatility $\sigma(S_t,t)$ is stochastic, Equation (A.12) cannot be solved analytically. However, the situation is a little more tractable if we assume the following: the volatility is a deterministic local volatility $\sigma(S_t,t)$ and both the risk-free interest rate and dividend yield are deterministic functions. Note that the local volatility $\sigma(S_t,t)$ should be defined such that it is locally *Lipschitz* and that the *Cauchy-Peano* local existence theorem[9] for ordinary differential equations holds (Duffie, 1996; Hassani, 1991).

To explain this we define $B(t)$ to be the value of a bank account at time $t \geq 0$. We assume $B(0) = 1$ and that the bank account evolves according to the following differential equation

$$dB(t) = r_t B(t) dt, \quad B(0) = 1$$

where r_t is a positive function of time (Brigo & Mercurio, 2001). If we integrate we get

$$B(t) = \exp\left(\int_0^t r_u du\right).$$

[9]Compare the Picard-Lindelöf theorem or Picard's existence theorem as well

Remember, r_u is the instantaneous rate at which the bank account accrues in a very short period du. Note that we integrate over $[0, t]$. Following from this we can define the stochastic discount factor $D(t, T)$ between t and T as follows

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_u du\right). \tag{A.13}$$

Here, $D(t, T)$ is the amount at time t that is equivalent to one unit of currency payable at time T.

If we now substitute (A.13) into (A.12) and we also assume our contingent claim is a vanilla option with a strike price of K, we have (we drop the subscript 0)

$$V_l(S,t) = D(t,T)E^{\mathbb{Q}}\left[\phi(S_T - K)^+ | S_t = S\right]. \tag{A.14}$$

However, if we assume our risk-free rates are given in continuous compounding format, we have $D(t, T) = \exp(-r_\tau \tau)$ where $\tau = T - t$ and thus

$$V_l(S,t) = e^{-r_\tau \tau}E^{\mathbb{Q}}\left[\phi(S_T - K)^+ | S_t = S\right]. \tag{A.15}$$

If $t = 0$ the rate r_τ is a zero coupon rate read off from a relevant yield curve. Otherwise r_τ is a relevant forward rate that holds from t to T and obtained from the zero-coupon yield curve rates for t and T. Here, ϕ is an indicator function: $\phi = 1$ for a call and $\phi = -1$ for a put.

A.2. Feynman-Kac in Integral Form

Equation (A.15) is the solution to the local volatility Black-Scholes PDE given in Equation (4.3). However, due to the expectation, it still seems difficult to solve. We also stated that we will use Monte Carlo simulation to solve the Black-Scholes PDE. Monte Carlo simulation is associated with integration. What now?

Remember that the fundamental law of mathematical expectation states: the expectation of a discrete random variable X is defined as

$$E(X) = \sum_{j=1}^{n} x_j f(x_j) \tag{A.16}$$

provided the sum is finite (Arnold, 1990). Here X is a discrete random variable having the possible values $x_1, x_2, \ldots, x_n$ with density function $f(x_j)$. We think of $E(X)$ as the average value of X — e.g., the average profit in a game of chance. A special case of Equation (A.16) is where all probabilities are equal such that

$$E(X) = \frac{1}{n}(x_1 + x_2 + \ldots + x_n).$$

This is of course the arithmetic mean. It acts as a representative or average of the values of X and is often called a *a measure of central tendency* (Spiegel et al., 2000).

For a continuous random variable X having density function $f(x)$, the expectation of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \tag{A.17}$$

where $x \in \mathbb{R}$ and provided the integral is finite or converges absolutely (Arnold, 1990).

Using the mathematical law of expectation, the expectation for a call option in Equation (A.15) can be written as an integral such that (Duffie, 1996; Wilmott, 2000)

$$V(S,t) = e^{-r_T(T-t)} \int_{K}^{\infty} \max[0, (S_T - K)] g(S_T) dS_T \tag{A.18}$$

where $g(S_T)$ is the probability density function (pdf) of S_T and we assume $\ln(S_T)$ is normally distributed with a standard deviation of w. We thus need to integrate over all possible S-values that is larger than the strike K at expiry. For a put we have

$$V(S,t) = e^{-r_T(T-t)} \int_{0}^{K} \max[0, (K - S_T)] g(S_T) dS_T. \tag{A.19}$$

Remember $K, S \in \mathbb{R}_0^+$

Under the assumption of a constant volatility and interest rate, the integral in Equation (A.18) can be solved analytically leading to the well-known Black-Scholes option pricing formulae for calls and puts. However, if we just relax the assumptions of constant volatility and constant interest rate slightly and assume that these two quantities are deterministic (but not constant), the integral cannot be calculated analytically anymore. The integral needs to be solved numerically.

The integral can be solved using Monte Carlo simulation. However, in order to do that, we need to know how S_T behaves or what the dynamics of S_T is. This is now quite simple because we know that equations (A.12) and (A.15) are only valid if the asset price dynamics are described by the stochastic differential equation given in (4.2).

A.3. Integrating the SDE

> BS option pricing: Along the way, it changed the way investors and others place a value on risk, giving rise to the field of risk management, the increased marketing of derivatives, and widespread changes in the valuation of corporate liabilities. The theory "is absolutely crucial to the valuation of anything from a company to property rights". Financial economics deals with four main phenomena: time, uncertainty, options and information.

The stochastic differential equation given in Equation (4.2) describes the dynamics of our stochastic asset price S. However, we understand this SDE much better if we make a change of variables. Remember, we stated that $\ln(S_T)$ is normally distributed so let's re-write (4.2) in terms of $\ln(S)$. Let's consider the process $X_t = f(S_t)$ defined by $f(x) = ln(x)$ (Clark, 2011). Remember that $f'(x) = 1/x$ and $f''(x) = -x^{-2}$. A simple application of Itô's lemma gives

$$dX_t = (r_t - d_t)dt + \sigma(S_t, t)dW_t - \frac{1}{2}\sigma^2(S_t, t)dt. \tag{A.20}$$

Remember, W_t is a standard Brownian motion and as such $dW_t = \varepsilon\sqrt{dt}$ where $\varepsilon \sim N(0,1)$, $N(0,1)$ being a standardised normal distribution. Following from this, Equation (A.20) can be integrated to give

$$X_T = X_0 + \left((r_T - d_T) - \frac{1}{2}\sigma^2(S_T, T)\right)T + \sigma(S_T, T)\varepsilon\sqrt{T}. \tag{A.21}$$

But, $X_t = \ln(S_t)$, thus

$$S_T = S_0 \exp\left(\left((r_T - d_T) - \frac{1}{2}\sigma^2(S_T, T)\right)T + \sigma(S_T, T)\varepsilon\sqrt{(T)}\right). \tag{A.22}$$

Equation (A.22) formulates a way to obtain the terminal value of the stochastic process S. This, together with Equation (A.18) can now be used to obtain the value of our option $V(S,t)$.

A.4. Discretising the SDE

Equations (4.2) and (A.21) are both defined for a continuous time variable t. So the question is how do we sample from the continuous distribution for the variable S_T? We do not have a mechanism for doing that. In order to model or simulate the security prices in practice we need to discretise the time in the process given in Equation (A.21). In this setting we partition $[0,T]$ into N equal subintervals of length Δt and we let (Jäckel, 2002; Hull, 2012)

$$\begin{aligned} dt &\approx \Delta t \\ \Delta t &= \frac{T}{N} \\ dS &\approx \Delta S = S_t - S_{t-1}. \end{aligned}$$

We then simulate S as a transition over each subinterval $[t, t+\Delta t]$ by using a discrete first order approximation. We call this an Euler approximation or Euler scheme. Under this first order approximation, Equation (A.21) can be written as follows (Glassermann, 2004)

$$S(t+\Delta t) = S(t)\exp\left[\left((r(t) - d(t)) - \frac{\sigma^2(S(t),t)}{2}\right)\Delta t + \sigma(S(t),t)\varepsilon_t\sqrt{\Delta t}\right]. \tag{A.23}$$

The Euler scheme is equivalent to approximating an integral using the left Riemann sum rule for approximating the value of an integral. Hence the integral is approximated as the product of the integrand at time t and the integration range dt. The diffusion term in the Euler scheme is $O(\sqrt{\Delta t})$ and it has strong convergence of order 1/2. This means we can always fall back on this workhorse of a numerical procedure to test any other method (Jäckel, 2002).

Equation (A.23) is called a difference equation meaning the asset price S at time $t + \Delta t$ is dependent on the price of S at a previous time t. Note, we need the price at a time T; $T \geq t$. S_T is obtained by incrementally stepping through time until we get to the N-th subinterval. We can explain this more clearly if we change subscripts in (A.23) to give

$$S(t_{i+1}) = S(t_i) \exp \left[\left((r(t_i) - d(t_i)) - \frac{\sigma^2(S(t_i), t_i)}{2} \right) \Delta t + \sigma(S(t_i), t_i)\, \varepsilon_t \sqrt{\Delta t} \right]. \tag{A.24}$$

Here, $i = 1, 2, \ldots, N$ such that $t_i = i\Delta t$ and $T = N\Delta t$. In order to start the simulation we need a starting asset value $S(t_0)$. If we then have the input parameters like the volatilities, risk-free rates and dividend yields, we can estimate a price for S at each discretised step i until we reach $S(t_N) = S(T)$.

The Euler scheme can be improved if we include the next order terms of the Itô-Taylor expansion of Equation (4.1). This gives (Jäckel, 2002; Glassermann, 2004; Clark, 2011)

$$S(t_{i+1}) = S(t_i) \exp \left[\left((r(t_i) - d(t_i)) - \frac{\sigma^2(S(t_i), t_i)}{2} \left[\varepsilon_t^2 - 1 \right] \right) \Delta t + \sigma(S(t_i), t_i)\, \varepsilon_t \sqrt{\Delta t} \right]. \tag{A.25}$$

By adding a term where the diffusion is $O(\Delta t)$ we get convergence of strong order 1. One of the advantages of Milstein over Euler time stepping is improved convergence when Δt is infinitesimal. In that case we can take larger time steps and get by with a smaller number of time steps N.

A.5. Now, Monte Carlo Simulation

In section A.2 we asked the question of how one can use Monte Carlo simulation in solving a PDE. We then explained that the solution to the Black-Scholes Equation (4.1) can be written in integral form as shown in Equation (A.15). Integrals can easily be evaluated by Monte Carlo simulation (Robert & Casella, 2004). The discretised version was given in Equation (A.16). If we now discretise equations (A.18) and (A.19) we have (Duffie, 1996; Glassermann, 2004; Jäckel, 2002)

$$V_{MC}(S, t) = e^{-r_T(T-t)} \frac{1}{M} \sum_{i=1}^{M} \max[0, \phi(S_T - K)] \tag{A.26}$$

where S_T is attained after N time steps that coincide with the expiry time T. We can use either Equation (A.24) or Equation (A.25) to estimate S_T. To obtain the Monte Carlo option price, we need to obtain M, S_T values. This means we simulate S_T, M times to obtain the average option value V_{MC}. Equation (A.26) is the simplest Monte Carlo approximation of the integral in equations (A.18) and (A.19). Note: N is the number of time steps and M the number of simulations.

B. ALSI Deterministic Volatility Function

Let's quickly summarise the deterministic linear but quadratic functional form for the ALSI implied volatility surface. The 3 dimensional market volatility surface is defined through the following function

$$\sigma_{imp}(S,K,t) = \sigma_{ATM}(t) + \frac{\theta_1}{t^{\lambda_1}}\left(\frac{K}{S}-1\right) + \frac{\theta_2}{t^{\lambda_2}}\left(\left(\frac{K}{S}\right)^2-1\right). \tag{B.27}$$

Safex obtains the ATM volatilities from the market meaning $\sigma_{ATM}(t)$ is a constant for every t. All the other parameters, θ_1, θ_2, λ_1 and λ_2 are obtained by optimising Equation (B.27) to the market traded data.

Safex publishes two other parameters: θ_A and λ_A. These parameters are the at-the-money parameters and give the theoretical ATM term structure of volatility. They are obtained by fitting

$$\sigma_{model}^{ATM}(t) = \frac{\theta_A}{t^{\lambda_A}} \simeq \sigma_{MtM}^{ATM}(t) \tag{B.28}$$

to the market (or mark-to-market (MtM)) term structure of ATM volatilities $\sigma_{MtM}^{ATM}(t)$ (see Kotzé & Joseph (2009)).

We need to mention a practical implementation point here. The term structure of ATM volatilities as obtained from the model in Equation (B.28) will not coincide with all the traded or mark-to-market ATM volatilities due to the numerical fitting procedure. We must, however, ensure that if we price an option expiring on a particular date, that $\sigma_{ATM}(t)$ in Equation (B.27) equates the market ATM volatility for that date. As an example, if we price a 9 month option ($T = 0.75$) and we have the 9 month mark-to-market ATM volatility $\sigma_{MtM}^{ATM}(0.75)$, we need to ensure that $\sigma_{ATM}(0.75)$) in Equation (B.27) is equal to this volatility. This is achieved by floating $\sigma_{model}^{ATM}(t)$ up or down by a constant amount such that $\sigma_{ATM}(0.75) = \sigma_{MtM}^{ATM}(0.75) = \sigma_{model}^{ATM}(0.75)$. This will in general have the effect that $\sigma_{ATM}(t)$ is not equal to the mark-to-market volatilities for $t \neq 0.75$.

The whole volatility surface is now described by a functional form given in Equation (B.27). The derivatives in Dupire's local volatility function in Equation (8.11) can be obtained analytically such that we have

$$\begin{aligned}
\frac{\partial \sigma_{imp}(S,K,\tau)}{\partial K} &= \frac{1}{S}\frac{\theta_1}{\tau^{\lambda_1}} + 2\frac{K}{S^2}\frac{\theta_2}{\tau^{\lambda_2}} \\
\frac{\partial^2 \sigma_{imp}(S,K,\tau)}{\partial K^2} &= 2\frac{1}{S^2}\frac{\theta_2}{\tau^{\lambda_2}} \\
\frac{\partial \sigma_{imp}(S,K,\tau)}{\partial \tau} &= -\lambda_1\theta_1\tau^{-(\lambda_1+1)}\left(\frac{K}{S}-1\right) - \lambda_2\theta_2\tau^{-(\lambda_2+1)}\left(\left(\frac{K}{S}\right)^2-1\right).
\end{aligned} \tag{B.29}$$

Using Equations (B.29) will lead to a smooth Dupire local volatility surface for $\sigma_{loc}(S_\tau,\tau)$) in Equation (8.11). Further, if the implied volatility surface in Equation

(B.27) is arbitrage-free, the corresponding Dupire local volatility surface should be arbitrage-free as well.

Let's look at a practical example. In Table 5 we show the parameter values, θ_i and λ_i, $(i = 1, 2, 3, ATM)$ as published by Safex on 28 May 2014. Also shown are the values for θ_i/t^{λ_i}, $i = 0, 1, 2, 3$. Table 6 lists σ_{imp}^{ATM}, the model ATM volatilities and official Safex ATM volatilities for all expiry dates. In Figure 4 we show the Alsi implied and corresponding local volatility surfaces.

		Curvature			
		Rho(θ_1)	VolVol (θ_2)	Level (θ_0)	Atm (θ_{ATM})
	In Months	-0.8488985	0.1945430	0.9139862	0.1350075
	In Years	-0.4337514	0.1069264	0.4753064	0.1595808
		Decay			
		Rho (λ_1)	VolVol (λ_2)	Level (λ_0)	Atm (λ_{ATM})
		0.2702186	0.2408592	0.2631310	-0.0672942
Date	T	θ_1/t^{λ_1}	θ_2/t^{λ_2}	θ_0/t^{λ_1}	
19-06-2014	0.06027397	-92.655786%	21.033029%	99.531201%	
18-09-2014	0.30958904	-59.544292%	14.181881%	64.708854%	
18-12-2014	0.55890411	-50.759237%	12.301016%	55.393271%	
19-03-2015	0.80821918	-45.943944%	11.255306%	50.269616%	
18-06-2015	1.05753425	-42.724414%	10.549535%	46.836131%	
17-09-2015	1.30684932	-40.349172%	10.025150%	44.298712%	
15-12-2016	2.55342466	-33.668883%	8.531503%	37.140432%	
21-12-2017	3.56986301	-30.754183%	7.869980%	34.005870%	

Table 5: Optimised parameters for the Alsi deterministic implied volatility function on 28 May 2014.

Expiry Date	**Expiry Time** T	**Model ATM** $\sigma_{model}^{ATM}(t)$	**Safex ATM** $\sigma_{MtM}^{ATM}(t)$
19-06-2014	0.06027397	13.209622%	14.25
18-09-2014	0.30958904	14.747329%	14.00
18-12-2014	0.55890411	15.345386%	14.50
19-03-2015	0.80821918	15.731053%	15.00
18-06-2015	1.05753425	16.018262%	15.75
17-09-2015	1.30684932	16.248072%	16.75
15-12-2016	2.55342466	16.997206%	18.50
21-12-2017	3.56986301	17.384842%	21.00

Table 6: Model and official Safex ATM volatilities on 28 May 2014.

C. Closed-Form Valuation of Single Barrier Options

Barrier options are standard calls and put except that they either disappear (the option is knocked out) or appear (the option is knocked in) if the underlying asset price breach a predetermined level (the barrier) (Haug, 2007). Barrier options are thus conditional options, dependent on whether the barriers have been crossed within the lives of the options. These options are also part of a class of options called path-dependent options[10].

Single barrier options are probably the oldest of all exotic options and have been traded sporadically in the US market since 1967 (Zhang, 1998). These options were developed to fill certain needs of hedge fund managers. Barrier options provided hedge funds with two features they could not obtain otherwise: the first is that most "down-and-out" options were written on more volatile stocks and these options are significantly cheaper than the corresponding vanilla options. The second feature is the increased convenience during a time when the trading volume of stock options was rather low. In other words, barrier options were created to provide risk managers with cheaper means to hedge their exposures without paying for price ranges that they believe unlikely to occur. Barrier options are also used by investors to gain exposure to (or enhance returns from) future market scenarios more complex than the simple bullish or bearish expectations embodied in standard options. The features just mentioned have helped to make barrier options the most popular path-dependent options being traded world wide.

We define two types of barriers: a barrier above the current asset price is an '*up barrier*'; if it is ever crossed it will be from below. A barrier below the current asset price is called a '*down barrier*'; if it is ever crossed it will be from above. Barrier options can also be divided into two classes: *in options* and *out options*.

C.1. Defining Single Barrier Payoffs

An *in barrier* (or *knock-in* option) will pay off only if the asset price finishes in-the-money and if the barrier is breached sometime before expiration. Every *knock-in* option starts inactive (it does not yet exist) and will stay inactive if the barrier is never crossed – in this situation the option expires worthless[11]. When the asset price crosses the barrier, the *in barrier* option is *knocked in* and becomes a standard vanilla option of the same type (call or put) with the payoff the same as a standard option.

An *out barrier* (or *knock-out* option) will pay out only if the asset price finishes in-the-money and the barrier is never breached before expiration – the payoff is the same as a standard option. Every *knock-out* option starts out as a standard vanilla option (call or put). Its behaviour is exactly the same as that of a vanilla option as long as the asset price never crosses the barrier. If the asset price crosses the barrier,

[10]A path-dependent option is an option whose payoff depends on the history of the underlying asset price. Other path-dependent options are Asian options, look-back options, ladder options and chooser or shout options.

[11]This means an investor buys an option that is worthless. This option will only be of any value when the barrier is crossed and the payoff is then the same as that of an ordinary vanilla option.

the option is *knocked out* and it expires worthless (the option becomes *null and void* and there is no chance of recovery).

There are eight types of vanilla barrier options:

1. up-and-out call and put,
2. up-and-in call and put,
3. down-and-out call and put,
4. down-and-in call and put.

Barrier options can also have cash rebates associated with them. This is a consolation prize paid to the holder of the option when an *out barrier* is knocked out or when an *in barrier* is never knocked in. The rebate can be nothing or it could be some fraction of the premium. Rebates are usually paid immediately when an option is knocked out, however, payments can be deferred to the maturity of the option.

The payoff functions for all barrier options are

$$V(\text{down-and-out}) = \begin{cases} \max\left[0, \phi\left(S-K\right)\right] & if\ S > H \text{ before expiry} \\ R & if\ S \leq H \text{ before expiry} \end{cases} \tag{C.30}$$

$$V(\text{up-and-out}) = \begin{cases} \max\left[0, \phi\left(S-K\right)\right] & if\ S < H \text{ before expiry} \\ R & if\ S\ \geq H \text{ before expiry} \end{cases} \tag{C.31}$$

$$V(\text{down-and-in}) = \begin{cases} \max\left[0, \phi\left(S-K\right)\right] & if\ S \leq H \text{ before expiry} \\ R & if\ S\ > H \text{ before expiry} \end{cases} \tag{C.32}$$

$$V(\text{up-and-in}) = \begin{cases} \max\left[0, \phi\left(S-K\right)\right] & if\ S \geq H \text{ before expiry} \\ R & if\ S\ < H \text{ before expiry} \end{cases} . \tag{C.33}$$

Here ϕ is the binary operator

$$\phi = \begin{cases} 1 & \text{for a call} \\ -1 & \text{for a put} \end{cases} .$$

Equation (C.30) shows that the payoff for a down-and-out barrier option is exactly the same as that for a vanilla option. However, we impose one extra boundary condition onto this payoff. That is that the option vanishes if the underlying's price breaches the barrier level H any time before expiry and the payout is the consolation prize of the rebate R. This one extra condition ensures that the option is path dependent and due to the skew, more difficult to price. It can however be done easily with a local volatility Monte Carlo scheme.

C.2. Closed-Form Solutions

Merton (1973) was first at deriving a closed-form solution for a barrier option where he showed that a European barrier option can be valued in a Black-Scholes environment — this means we have a fixed volatility and interest rate and dividend yield. Thereafter, Rubinstein & Reiner (1991) generalised barrier option-pricing theory. Rich (1994) gives an excellent summary of barrier options. Broadie et al. (1997) gives a simple modification to adjust the prices if the barrier is monitored discretely in time e.g., daily or weekly. With a rebate, continuous dividend yield and continuous monitoring of the barrier, the following equations are obtained (Haug, 2007):

$$\left.\begin{array}{l}
A = \phi S\, e^{-d\tau} \left(\frac{H}{S}\right)^{2\lambda} N\left(\eta y\right) - \phi K\, e^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y - \eta\sigma\sqrt{\tau}\right) \\
B = R\, e^{-r\tau} \left[N\left(\eta x_1 - \eta\sigma\sqrt{\tau}\right) - \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y_1 - \eta\sigma\sqrt{\tau}\right)\right] \\
C = \phi S\, e^{-d\tau} N\left(\phi x\right) - \phi K\, e^{-r\tau} N\left(\phi x - \phi\sigma\sqrt{\iota}\right) \\
D = \phi S\, e^{-d\tau} N\left(\phi x_1\right) - \phi K\, e^{-r\tau} N\left(\phi x_1 - \phi\sigma\sqrt{\tau}\right) \\
E = \phi S\, e^{-d\tau} \left(\frac{H}{S}\right)^{2\lambda} N\left(\eta y_1\right) - \phi K\, e^{-r\tau} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y_1 - \eta\sigma\sqrt{\tau}\right) \\
F = R\left[\left(\frac{H}{S}\right)^{a+b} N\left(\eta z\right) + \left(\frac{H}{S}\right)^{a-b} N\left(\eta z - 2\eta b\sigma\sqrt{\tau}\right)\right]
\end{array}\right\} \tag{C.34}$$

where S is the spot market price, K is the strike price, H is the barrier (in the same units as S and K), R is the rebate (in currency units), τ is the annualised time till expiration, r is the risk-free short term interest rate in continuous format, d is the dividend yield in continuous format, σ is the volatility and ϕ and η are binary variables set out in Table C.2.

All the other variables are defined as follows (with ln the natural logarithm)

$$\left.\begin{array}{l}
x = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{S}{K}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\} \\
x_1 = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{S}{H}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\} \\
y = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H^2}{SK}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\} \\
y_1 = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H}{S}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\} \\
z = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H}{S}\right) + b\sigma^2\tau\right\} \\
\lambda = 1 + \frac{\mu}{\sigma^2} \\
a = \frac{\mu}{\sigma^2} \\
b = \frac{1}{\sigma^2}\left[\sqrt{\mu^2 + 2r\sigma^2}\right] \\
\mu = r - d - \frac{\sigma^2}{2}.
\end{array}\right\} \tag{C.35}$$

$N(\bullet)$ is the cumulative of the normal distribution function (Haug, 2007; Hull, 2012).

The valuation formulas for the eight barrier options can be written as combinations of the quantities A to F given in Equation (C.34). The value of each barrier is also dependent on whether the barrier H is above or below the strike price K. All barriers are priced in using equations (C.34) and (C.35) and combining them as shown in Table C.2.

The abbreviations used are: $DIC_{K<H}$ is short for “down and in call barrier option” where the strike value $K$is less than the barrier value H. If the payment of the rebate is deferred to maturity for the *knock-out* options, we put $F = B$ in the equations above.

Call	Put
Down and In Barriers	
$\phi = \eta = 1$ $DIC_{K \geq H} = A + B$ $DIC_{K<H} = C - D + E + B$	$\phi = -1, \eta = 1$ $DIP_{K \geq H} = D - A + E + B$ $DIP_{K<H} = C + B$
Up and In Barriers	
$\phi = 1, \eta = -1$ $UIC_{K \geq H} = C + B$ $UIC_{K<H} = D - A + E + B$	$\phi = -1, \eta = -1$ $UIP_{K \geq H} = C - D + E + B$ $UIP_{K<H} = A + B$
Down and Out Barriers	
$\phi = \eta = 1$ $DOC_{K \geq H} = C - A + F$ $DOC_{K<H} = D - E + F$	$\phi = -1, \eta = 1$ $DOP_{K \geq H} = C - D + A - E + F$ $DOP_{K<H} = F$
Up and Out Barriers	
$\phi = 1, \eta = -1$ $UOC_{K \geq H} = F$ $UOC_{K<H} = C - D + A - E + F$	$\phi = -1, \eta = -1$ $UOP_{K \geq H} = D - E + F$ $UOP_{K<H} = C - A + F$

Table 7: Pricing Formulas for European barrier options. The variables are defined in Equation (C.34)

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